

# On the Critical Behavior of Continuous Homopolymers

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## Abstract

The aim of this paper is to investigate the distribution of a continuous homopolymer in the presence of an attractive finitely supported potential. The most intricate behavior can be observed if we simultaneously vary two parameters: the temperature, which approaches the critical value, and the length of the polymer, which tends to infinity. As the main result, we identify the distributions that appear in the limit (after a diffusive scaling of the original polymer measures) and depend on the relation between the two parameters.

## 1 Introduction

We consider the following model of long homogeneous polymer chains in an attractive potential field. Let the space  $C([0, T], \mathbb{R}^d)$  be equipped with the Wiener measure  $P_{0,T}$  and suppose that we have a smooth, nonnegative, not identically equal to zero, compactly supported potential  $v \in C_0^\infty(\mathbb{R}^d)$  and a coupling constant  $\beta \geq 0$  (inverse temperature), which regulates the strength of the attraction. The elements  $\omega(\cdot)$  of the space are interpreted as realizations of a continuous homopolymer on  $[0, T]$  and are distributed according to the Gibbs measure  $P_{\beta,T}$  with

$$\frac{dP_{\beta,T}}{dP_{0,T}}(\omega) = \frac{e^{\beta \int_0^T v(\omega(t))dt}}{Z_{\beta,T}}, \quad \omega \in C([0, T], \mathbb{R}),$$

where

$$Z_{\beta,T} = E_{0,T} e^{\beta \int_0^T v(\omega(t))dt}$$

is the partition function.

Thus the polymer measure we consider is of a “mean field” type, where the polymer chain interacts with the external attractive potential (as in [6]). While the potential is assumed to be constant in time in our paper, many interesting results have been obtained for disordered media, i.e., time-dependent random potentials (existence of phase transitions, dependence of the growth rate of the partition function on the temperature, etc., see [2], [5], for example). We suspect, however, that the disordered models are not

amenable to such a detailed analysis of the phase transition phenomena as found in the current paper.

It is established - using the Feynman-Kac formula - that under the measures  $P_{\beta,T}$ , the processes  $\{\omega(t), 0 \leq t \leq T\}$  are time-inhomogeneous and Markovian and that their transition densities can be expressed in terms of the fundamental solution  $p_\beta$  of the parabolic equation

$$\frac{\partial u}{\partial t} = H_\beta u, \quad \text{where} \quad H_\beta = \frac{1}{2}\Delta + \beta v : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d). \quad (1)$$

More precisely, the finite-dimensional distributions are

$$\begin{aligned} P_{\beta,T}(\omega(t_1) \in A_1, \dots, \omega(t_n) \in A_n) &= \\ &= \frac{1}{Z_{\beta,T}} \int_{A_1} \dots \int_{A_n} \int_{\mathbb{R}^d} p_\beta(t_1, 0, x_1) \dots p_\beta(T - t_n, x_n, y) dy dx_n \dots dx_1 \end{aligned} \quad (2)$$

for  $0 \leq t_1 \leq \dots \leq t_n \leq T$  and  $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^3)$ .

It is worth noting that

$$Z_{\beta,t} = \int_{\mathbb{R}^3} p_\beta(t, 0, y) dy.$$

It was shown that in  $d \geq 3$ , at a certain (critical) value of the coupling constant, there occurs a phase transition between a densely packed globular state and an extended phase of the polymer. Namely, for  $\beta > \beta_{\text{cr}}$ , a typical polymer realization is at a distance of order one from the origin as  $T \rightarrow \infty$ . On the other hand, for  $\beta \leq \beta_{\text{cr}}$ , the realizations need to be scaled by the factor  $\sqrt{T}$  in the spatial variables in order to get a non-trivial limit (as in Theorem 1.1 below). The critical value of the coupling constant is

$$\beta_{\text{cr}} = \sup\{\beta > 0 \mid \sup \sigma(H_\beta) = 0\},$$

where  $\sigma(H_\beta)$  is the spectrum of the operator  $H_\beta$ .

Here's the precise statement of the result proved in [3] that is relevant to this paper.

**Theorem 1.1.** *For  $a > 0$ , let  $f_a : C([0, T], \mathbb{R}^3) \rightarrow C([0, aT], \mathbb{R}^3)$  be the mapping defined by  $(f_a \omega)(t) = \sqrt{a} \omega(t/a)$ , and let  $f_a P$  be the push-forward of a measure  $P$  by this mapping. There is a measure  $Q_0$  on  $C([0, 1], \mathbb{R}^3)$  that corresponds to a certain time-inhomogeneous Markov process starting at the origin such that*

$$f_{T^{-1}} P_{\beta_{\text{cr}}, T} \Rightarrow Q_0 \quad \text{as } T \rightarrow \infty.$$

**Remark 1.2.** *We use the subscript 0 in the notation for the limiting measure since we will introduce a whole class of measures, and  $Q_0$  will be just a particular member of this class.*

**Remark 1.3.** *For  $\beta < \beta_{\text{cr}}$ , the measures  $f_{T^{-1}} P_{\beta, T}$  converge to the Wiener measure on  $C([0, 1], \mathbb{R}^3)$ . Note, however, that this convergence and the convergence in Theorem 1.1 take place for fixed values of  $\beta$  that don't scale with  $T$ .*

In this paper we'll allow  $\beta = \beta(T)$  to change in such a way that  $(\beta(T) - \beta_{\text{cr}})\sqrt{T} \rightarrow \chi$ . We'll show that there are measures  $Q_\gamma$  and a linear mapping  $\gamma(\chi) = c\chi$  such that

$$f_{T^{-1}}P_{\beta(T),T} \Rightarrow Q_{\gamma(\chi)} \quad \text{as } T \rightarrow \infty.$$

Before formulating this as a theorem, let us describe the limiting measures  $Q_\gamma$ . They were introduced in [4] as the polymer measures on  $C([0,1], \mathbb{R}^3)$  corresponding to zero-range attracting potentials (i.e., the potentials that are, roughly speaking, concentrated at the origin). More precisely, it was shown in [4] that, for each  $\gamma \in \mathbb{R}$ , the polymer measures corresponding to the potentials

$$v_\gamma^\varepsilon(x) = \left(\frac{\pi^2}{8\varepsilon^2} + \frac{\gamma}{\varepsilon}\right)\chi_B\left(\frac{x}{\varepsilon}\right) \quad (3)$$

converge as  $\varepsilon \downarrow 0$ . (Here  $\chi_B$  is the indicator function of the unit ball centered at the origin.) The limit will be denoted by  $Q_\gamma$ . It is worth noting that all the measures  $Q_\gamma$  are spherically symmetric and thus the dependence of the polymer measures  $P_{\beta(T),T}$  on the 'shape' of the potential disappears in the limit.

Another, less intuitive, but more convenient, way to define the measures  $Q_\gamma$  is through the self-adjoint extensions of the Laplacian acting on  $\mathbb{R}^3 \setminus 0$ . Namely, it was shown in [1] that there is a one-parameter family  $\{\mathcal{L}_\gamma, \gamma \in \mathbb{R}\}$  of self-adjoint operators acting on  $L^2(\mathbb{R}^3)$  such that  $\mathcal{L}_\gamma f = \Delta f$  whenever  $f \in C_0^\infty(\mathbb{R}^3 \setminus 0)$ . The kernel of  $\exp(t\mathcal{L}_\gamma)$ ,  $t > 0$ , is given by

$$\bar{p}_\gamma(t, x, y) = \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{3/2}} + \frac{1}{4\pi^2 i} \int_{\Gamma(a)} \frac{e^{-\sqrt{2\lambda}(|x|+|y|)+\lambda t}}{(\sqrt{2\lambda} - \gamma)|x||y|} d\lambda, \quad (4)$$

where  $x, y \neq 0$ ,  $\Gamma(a) = \{z \in \mathbb{C} | \operatorname{Re} z = a\}$ , and  $a > \gamma^2/2$ . Thus  $\bar{p}_\gamma(t, x, y)$  can be interpreted as the formal fundamental solution of the parabolic equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\gamma u.$$

By analogy with (2), we can define the measures  $\bar{P}_{\gamma,T}^x$ , whose finite-dimensional distributions are given by the formula

$$\begin{aligned} \bar{P}_{\gamma,T}^x(\omega(t_1) \in A_1, \dots, \omega(t_n) \in A_n) &= \\ &= \frac{1}{\bar{Z}_{\gamma,T}(x)} \int_{A_1} \dots \int_{A_n} \int_{\mathbb{R}^d} \bar{p}_\gamma(t_1, x, x_1) \dots \bar{p}_\gamma(T - t_n, x_n, y) dy dx_n \dots dx_1 \end{aligned}$$

for  $0 \leq t_1 \leq \dots \leq t_n \leq T$  and  $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^3)$ , where

$$\bar{Z}_{\gamma,T}(x) = \int_{\mathbb{R}^3} \bar{p}_\gamma(t, x, y) dy = 1 + \frac{1}{2\pi i} \int_{\Gamma(a)} \frac{e^{\lambda t}}{\sqrt{2\lambda} - \gamma} \frac{e^{-\sqrt{2\lambda}|x|}}{\lambda|x|} d\lambda.$$

Note the dependence of the measures on the initial point  $x \in \mathbb{R}^3$ . In fact, neither  $\bar{p}_\gamma(t_1, x, y)$  nor  $\bar{Z}_{\gamma,T}(x)$  are defined for  $x = 0$ , but we can make sense out of  $\bar{P}_{\gamma,T}^0$  by

taking the limit of  $\overline{P}_{\gamma,T}^x$  as  $x \rightarrow 0$ . If we put  $Q_\gamma = \overline{P}_{\gamma,1}^0$ , we'll obtain the same family of measures corresponding to zero-range potentials.

It is also worth mentioning that the assumption  $d = 3$  is important. Indeed, there is no phase transition for  $d = 1, 2$ , while for dimensions higher than 3 there are no nontrivial self-adjoint extensions of the Laplacian.

## 2 The main result

It was shown in Lemmas 5.4 and 5.5 of [3] that the solution space of the problem

$$\frac{1}{2}\Delta\psi + \beta_{\text{cr}}v(x)\psi = 0, \quad \psi(x) = \mathcal{O}(|x|^{-1}), \quad \frac{\partial\psi}{\partial r}(x) = \mathcal{O}(|x|^{-2}) \quad \text{as } |x| \rightarrow \infty$$

is one-dimensional, and  $\psi$  can be chosen to be positive.

The main result of our paper is the following.

**Theorem 2.1.** *If  $(\beta(T) - \beta_{\text{cr}})\sqrt{T} \rightarrow \chi$ , then*

$$f_{T^{-1}}P_{\beta(T),T} \Rightarrow Q_{\gamma(\chi)} \quad \text{as } T \rightarrow \infty,$$

where  $\gamma(\chi) = c\chi$  with

$$c = \sqrt{2}/(\beta_{\text{cr}}^2\gamma_1) \quad \text{and} \quad \gamma_1 = \frac{(\int_{\mathbb{R}^3} v(x)\psi(x)dx)^2}{\sqrt{2}\pi \int_{\mathbb{R}^3} v(x)\psi^2(x)dx}.$$

The proof will rely on the asymptotic formulas for the fundamental solution of (1) and the partition function, which can be found in the right-hand side of (2). We formulate these asymptotic formulas in several propositions below.

Let us introduce the space

$$C_{\text{exp}}(\mathbb{R}^3) = \left\{ f \in C(\mathbb{R}) \left| \|f\|_{C_{\text{exp}}} := \sup_{x \in \mathbb{R}^3} (|f(x)|e^{|x|^2}) < \infty \right. \right\}$$

For  $f \in C_{\text{exp}}(\mathbb{R}^3)$ , the solution of the parabolic problem

$$\frac{\partial u}{\partial t} = H_{\beta(T)}u, \quad u(0, x) = f(x) \tag{5}$$

is given by the inverse Laplace transform

$$u_{\beta(T)}(t, x) = -\frac{1}{2\pi i} \int_{\Gamma(\lambda_0(\beta(T)) + \delta/T)} e^{\lambda t} (R_{\beta(T)}(\lambda)f)(x) d\lambda, \tag{6}$$

where  $\lambda_0(\beta) = \sup \sigma(H_\beta)$ ,  $\delta > 0$ , and  $R_\beta(\lambda) = (H_\beta - \lambda I)^{-1}$  is the resolvent of  $H_\beta$ .

**Proposition 2.2.** *For  $\epsilon > 0$ , we have*

1. (The solution of the Cauchy problem)

$$u_{\beta(T)}(t, x) = \frac{1}{\sqrt{T}} \frac{\alpha(f)}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\frac{\lambda t}{T}}}{\sqrt{2\lambda} - \gamma} \frac{e^{-\sqrt{2\lambda} \frac{|x|}{\sqrt{T}}}}{|x|} d\lambda + q^f(T, t, x) \quad (7)$$

with  $\gamma = \sqrt{2}\chi/(\gamma_1\beta_{\text{cr}}^2)$ ,  $\delta > 0$ , and

$$\alpha(f) = \kappa \int_{\mathbb{R}^3} \psi(x) f(x) dx, \quad \kappa = \frac{1}{\beta_{\text{cr}} \int_{\mathbb{R}^3} v(x) \psi(x) dx},$$

where the error term satisfies

$$\sup_{\epsilon T \leq t \leq T, \epsilon \sqrt{T} \leq |x| \leq \epsilon^{-1} \sqrt{T}} |q^f(T, t, x)| = \|f\|_{C_{\text{exp}}} \mathcal{O}(T^{-3/2}), \quad (8)$$

2. (The fundamental solution)

$$p_{\beta(T)}(t, y, x) = \frac{1}{\sqrt{T}} \frac{\kappa \psi(y)}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\frac{\lambda t}{T}}}{\sqrt{2\lambda} - \gamma} \frac{e^{-\sqrt{2\lambda} \frac{|x|}{\sqrt{T}}}}{|x|} d\lambda + q(T, t, y, x), \quad (9)$$

where

$$\sup_{\epsilon T \leq t \leq T, \epsilon \sqrt{T} \leq |x| \leq \epsilon^{-1} \sqrt{T}, |y| \leq \epsilon^{-1}} T |q(T, t, y, x)| \rightarrow 0.$$

We'll also need the asymptotics of the fundamental solution when  $y$  is of order  $\sqrt{T}$ .

**Proposition 2.3.** For  $\epsilon > 0$ ,

$$p_{\beta(T)}(t, y, x) = p_0(t, y, x) + \frac{1}{\sqrt{T}} \frac{1}{4\pi^2 i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\frac{\lambda t}{T} - \sqrt{2\lambda} \frac{|x| + |y|}{\sqrt{T}}}}{(\sqrt{2\lambda} - \gamma) |x| |y|} d\lambda + q(T, t, y, x), \quad (10)$$

where  $p_0$  is the fundamental solution of the heat equation and

$$\sup_{\epsilon T \leq t \leq T, \epsilon \sqrt{T} \leq |y|, |x| \leq \epsilon^{-1} \sqrt{T}} T^{3/2} q(T, t, y, x) \rightarrow 0. \quad (11)$$

We will need one more result concerning the behavior of the partition sum with respect to the measure  $P_{0,T}^x$  (the Wiener measure  $P_{0,T}$  translated by the vector  $x$ ).

**Proposition 2.4.** If  $Z_{\beta,t}(x) = \mathbb{E}_{0,T}^x e^{\beta \int_0^T v(\omega(t)) dt}$ , then for every  $\epsilon > 0$ , we have

$$Z_{\beta(T),t}(x) = 1 + \frac{\sqrt{T}}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\frac{\lambda t}{T}}}{\sqrt{2\lambda} - \gamma} \frac{e^{-\sqrt{2\lambda} \frac{|x|}{\sqrt{T}}}}{\lambda |x|} d\lambda + q^Z(T, t, x),$$

where

$$\lim_{T \rightarrow \infty} \sup_{\epsilon \sqrt{T} \leq |x| \leq \epsilon^{-1} \sqrt{T}, \epsilon T \leq t \leq T} |q^Z(T, t, x)| = 0.$$

### 3 Proofs

Note that the expression (6) for the solution of (5) contains the resolvent of the operator  $H_\beta$  inside the integral. It will be seen that the main contribution to the integral comes from the values of  $\lambda$  that are close to zero. Therefore, it is important to know the asymptotics of the resolvent as  $\lambda \rightarrow 0$ . First, let us make several observations concerning the spectrum of  $H_\beta$  (a more detailed discussion of the spectral properties of  $H_\beta$  together with the proofs of the following four lemmas can be found in [3]).

It is well known that for some  $0 \leq N < \infty$

$$\sigma(H_\beta) = (-\infty, 0] \cup \{\lambda_j\}_{j=0}^N,$$

where the eigenvalues are enumerated in decreasing order.

**Lemma 3.1.** *For  $\beta \leq \beta_{\text{cr}}$ , we have  $\sup \sigma(H_\beta) = 0$ , while  $\sup \sigma(H_\beta) = \lambda_0(\beta) > 0$  for  $\beta > \beta_{\text{cr}}$ . In the latter case,  $\lambda_0(\beta)$  is a simple eigenvalue. It is a strictly increasing and continuous function of  $\beta$ . Moreover,  $\lim_{\beta \downarrow \beta_{\text{cr}}} \lambda_0(\beta) = 0$  and  $\lim_{\beta \uparrow \infty} \lambda(\beta) = \infty$ .*

Due to the monotonicity and continuity of  $\lambda = \lambda_0(\beta)$  for  $\beta > \beta_{\text{cr}}$ , we can define the inverse function

$$\beta = \beta_0(\lambda) : [0, \infty) \rightarrow [\beta_{\text{cr}}, \infty).$$

Let  $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$ .

The resolvent  $R_\beta(\lambda) = (H_\beta - \lambda I)^{-1}$  is a meromorphic operator valued function on  $\mathbb{C}'$ . Let us introduce the operator

$$A(\lambda) = v(x)R_0(\lambda) : C_{\text{exp}}(\mathbb{R}^d) \rightarrow C_{\text{exp}}(\mathbb{R}^d), \quad \lambda \in \mathbb{C}'.$$

We have the following identity for the resolvent

$$R_\beta(\lambda) = R_0(\lambda)(I + \beta A(\lambda))^{-1}, \lambda \in \mathbb{C}'. \quad (12)$$

From this it is not difficult to show that  $1/\beta_0(\lambda)$  is the principal eigenvalue of  $-A(\lambda)$  and using this, we can extend the domain of  $\beta_0$  to  $[0, \infty) \cup (U \cap \mathbb{C}')$ , where  $U$  is a sufficiently small neighborhood of zero.

**Lemma 3.2.** *(Asymptotic behavior)*

a) *The principal eigenvalue has the following behavior as  $\beta \downarrow \beta_{\text{cr}}$*

$$\lambda_0(\beta) = \frac{1}{\gamma^2 \beta_{\text{cr}}^4} (\beta - \beta_{\text{cr}})^2 (1 + o(1)).$$

b) *The small  $\lambda$  asymptotics of  $\beta_0(\lambda)$  is given by*

$$\frac{1}{\beta_0(\lambda)} = \frac{1}{\beta_{\text{cr}}} - \gamma_1 \sqrt{\lambda} + \mathcal{O}(\lambda), \quad \lambda \rightarrow 0, \lambda \in \mathbb{C}'. \quad (13)$$

**Lemma 3.3.** *We have*

- a) *For every  $\epsilon > 0$ ,  $R_0(\lambda) : C_{\text{exp}}(\mathbb{R}^3) \rightarrow C(\mathbb{R}^3)$  is a bounded operator with  $\|R_0(\lambda)\| = \mathcal{O}(|\lambda|^{-1})$  as  $\lambda \rightarrow \infty$ ,  $|\arg \lambda| \leq \pi - \epsilon$ .*
- b) *The operator  $(I + \beta A(\lambda))^{-1}$  on  $C_{\text{exp}}(\mathbb{R}^3)$  is meromorphic in  $\mathbb{C}'$ . For each  $\epsilon, \Lambda > 0$ , there is a  $\tilde{\delta} > 0$  such that it is uniformly bounded operator in  $\lambda \in \mathbb{C}'$ ,  $|\arg \lambda| \leq \pi - \epsilon, |\lambda| \geq \Lambda$ ,  $|\beta - \beta_{\text{cr}}| \leq \tilde{\delta}$ .*

As for the  $\lambda \rightarrow 0$  asymptotics,

**Lemma 3.4.** *There are  $\lambda_0 > 0$  and  $\delta_0 > 0$  such that for  $\lambda \in \mathbb{C}' \cup \{0\}$ ,  $|\lambda| \leq \lambda_0$ ,  $|\beta - \beta_{\text{cr}}| \leq \delta_0$ ,  $\beta \neq \beta_0(\lambda)$ , we have*

$$(I + \beta A(\lambda))^{-1} = \frac{\beta_0(\lambda)}{\beta_0(\lambda) - \beta} (B + S(\lambda)) + C(\lambda, \beta)$$

as operators on  $C_{\text{exp}}(\mathbb{R}^3)$ , where  $B$  is the one dimensional operator with kernel

$$B(x, y) = \frac{v(x)\psi(x)\psi(y)}{\int_{\mathbb{R}^3} v(x)\psi^2(x)dx},$$

$S = \mathcal{O}(\sqrt{|\lambda|})$ ,  $S(0) = 0$ , and  $C(\lambda, \beta)$  is bounded uniformly in  $\lambda$  and  $\beta$ .

Now for  $f \in C_{\text{exp}}(\mathbb{R}^3)$ , define

$$g_T^f(z, y) = (I + \beta(T)A(z))^{-1}f(y)$$

The key to the proofs in this paper is the following lemma.

**Lemma 3.5.** *For  $\delta > 0$ ,*

$$g_T^f\left(\frac{\lambda}{T}, y\right) = \sqrt{T} \frac{2\pi \int_{\mathbb{R}^3} \psi(x)f(x)dx}{\beta_{\text{cr}} \left(\int_{\mathbb{R}^3} v(x)\psi(x)dx\right)^2} \frac{v(y)\psi(y)}{\sqrt{2\lambda - \gamma}} + (K(\lambda, T)f)(y) \quad (14)$$

as  $T \rightarrow \infty$  for  $\lambda \in \Gamma(\gamma^2/2 + \delta)$ , where  $K(\lambda, T)$  is uniformly bounded as an operator on  $C_{\text{exp}}(\mathbb{R}^3)$ .

*Proof.* First consider the  $|\lambda| < aT$  case for  $a$  sufficiently small in order for the following to make sense. Lemma 3.2 yields

$$\beta_0(z) = \beta_{\text{cr}} + \beta_{\text{cr}}^2 \gamma_1 \sqrt{z} + \mathcal{O}(z), \quad z \rightarrow 0, z \in \mathbb{C}'. \quad (15)$$

Using that

$$\beta(T) = \beta_{\text{cr}} + \frac{\chi}{\sqrt{T}} + o\left(\frac{1}{\sqrt{T}}\right), \quad T \rightarrow \infty, \quad (16)$$

we can write for  $\lambda \in \Gamma(\gamma^2/2 + \delta)$  that

$$\beta_0\left(\frac{\lambda}{T}\right) - \beta(T) = \beta_{\text{cr}}^2 \gamma_1 \sqrt{\frac{\lambda}{T}} - \frac{\chi}{\sqrt{T}} + \mathcal{O}\left(\frac{\lambda}{T}\right) + o(1/\sqrt{T}) \quad (17)$$

as  $T \rightarrow \infty$ . Using the definition of  $\gamma$ , this can be rewritten as

$$\begin{aligned} \frac{\beta_0\left(\frac{\lambda}{T}\right)}{\beta_0\left(\frac{\lambda}{T}\right) - \beta(T)} &= \frac{\beta_0\left(\frac{\lambda}{T}\right)}{\frac{\beta_{\text{cr}}^2 \gamma_1}{\sqrt{2T}}(\sqrt{2\lambda} - \gamma) + \mathcal{O}\left(\frac{\lambda}{T}\right) + o\left(\frac{1}{\sqrt{T}}\right)} = \\ &= \frac{\sqrt{2T}}{\beta_{\text{cr}} \gamma_1 (\sqrt{2\lambda} - \gamma)} + \mathcal{O}_a(1), \end{aligned} \quad (18)$$

where  $\mathcal{O}_a(1)$  denotes a bounded quantity depending on  $a$ .

Combining (18) and Lemma 3.4 gives

$$g_T^f\left(\frac{\lambda}{T}, y\right) = \sqrt{T} \frac{2\pi \int_{\mathbb{R}^3} \psi(x) f(x) dx}{\beta_{\text{cr}} \left(\int_{\mathbb{R}^3} v(x) \psi(x) dx\right)^2} \frac{v(y) \psi(y)}{\sqrt{2\lambda} - \gamma} + \tilde{\mathcal{O}}_a(1) f,$$

as  $|\lambda| \leq aT$  and  $T \rightarrow \infty$ , where  $\tilde{\mathcal{O}}_a(1)$  is a bounded operator.

For  $|\lambda| > aT$ , the first term on the right hand side of (14) is uniformly bounded. By Lemma 3.3 and  $\beta(T) \rightarrow \beta_{\text{cr}}$ , so is the left hand side, and therefore their difference too.  $\square$

*Proof of Proposition 2.2.* Fix  $\delta > 0$  and note that

$$\begin{aligned} u_{\beta(T)}(t, x) &= \frac{1}{2\pi i} \int_{\Gamma(\lambda_0(\beta(T)) + \delta/T)} e^{\lambda t} (R_0(\lambda) g_T^f(\lambda, \cdot))(x) d\lambda = \\ &= \frac{1}{2\pi i} \int_{\Gamma(T\lambda_0(\beta(T)) + \delta)} e^{\lambda \frac{t}{T}} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{2\lambda} \frac{|x-y|}{\sqrt{T}}}}{2\pi|x-y|} g_T^f\left(\frac{\lambda}{T}, y\right) dy \frac{d\lambda}{T} \end{aligned} \quad (19)$$

after a change of variables, where we used the explicit expression for the kernel of  $R_0(\lambda)$ , which is just Green's function for the Laplacian on the whole space. Note that by Lemma 3.3 a), moving the contour is permitted.

By Lemma 3.2 and (16), we have

$$\lambda_0(\beta(T)) = \frac{1}{\gamma^2 \beta_{\text{cr}}^4} (\beta(T) - \beta_{\text{cr}})^2 + o((\beta(T) - \beta_{\text{cr}})^2) = \frac{\chi^2}{\gamma_1^2 \beta_{\text{cr}}^4} \frac{1}{T} + o\left(\frac{1}{T}\right), \quad (20)$$

and we get  $T\lambda_0(\beta(T)) \rightarrow \frac{\chi^2}{\gamma_1 \beta_{\text{cr}}^4} = \gamma^2/2$ . Therefore, for large enough  $T$ , we can take the path  $\Gamma(\gamma^2/2 + \delta)$  as the contour of integration.

Let us denote the first term in (14) by  $g_T^0$ , and let  $g_T^1 = g_T - g_T^0$ . If  $u_{\beta(T)}^0$  stands for the contour integral (19) with  $g_T^0$  in place of  $g_T$ , then we have

$$\begin{aligned} u_{\beta(T)}^0(t, x) &= \\ &= \frac{1}{\sqrt{T}} \frac{\alpha(f)}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\lambda \frac{t}{T}}}{\sqrt{2\lambda} - \gamma} \frac{1}{\int_{\mathbb{R}^3} v(w) \psi(w) dw} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{2\lambda} \frac{|x-y|}{\sqrt{T}}}}{|x-y|} v(y) \psi(y) dy d\lambda. \end{aligned} \quad (21)$$



It's easy to see by Taylor's formula that for  $y$  bounded,  $\epsilon\sqrt{T} \leq |x| \leq \epsilon^{-1}\sqrt{T}$ ,  $\lambda \in \Gamma(\gamma^2/2 + \delta)$ , there are  $C, \alpha > 0$  such that for large enough  $T$ ,

$$\left| \frac{e^{-\sqrt{2\lambda}\frac{|x-y|}{\sqrt{T}}}}{|x-y|} - \frac{e^{-\sqrt{2\lambda}\frac{|x|}{\sqrt{T}}}}{|x|} \right| \leq C \frac{e^{-\alpha\sqrt{\lambda}}}{T}$$

Plugging this back into (21), the first term gives the main term of (7), while the remainder is easily shown to satisfy (8) (as  $v$  has compact support, all integrals exist).

The remaining error term is

$$u_{\beta(T)}^1(t, x) = \frac{1}{2\pi iT} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} e^{\lambda \frac{t}{T}} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{2\lambda}\frac{|x-y|}{\sqrt{T}}}}{2\pi|x-y|} (K(\lambda, T)f)(y) dy d\lambda.$$

Splitting the spatial integration as

$$u_{\beta(T)}^1(t, x) = \frac{1}{2\pi iT} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \int_{\{|y| \leq \frac{\epsilon\sqrt{T}}{2}\}} + \int_{\{|y| > \frac{\epsilon\sqrt{T}}{2}\}} = I_a + I_b,$$

we get, after making the substitution  $\xi := \lambda t/T$ ,

$$I_a = \frac{1}{2\pi i} \frac{1}{T} \int_{\Gamma(\frac{t}{T}(\gamma^2/2 + \delta))} \int_{|y| \leq \frac{\epsilon\sqrt{T}}{2}} e^{\xi - \sqrt{2\xi}\frac{|x-y|}{\sqrt{t}}} \frac{1}{2\pi|x-y|} (K(\lambda(\xi), T)f)(y) e^{y^2} e^{-y^2} dy d\xi.$$

Now  $1/|x-y| < 2/(\epsilon\sqrt{T})$  yields

$$|I_a| \leq \frac{\|f\|_{C_{\text{exp}}} C(\epsilon)}{T^{3/2}}.$$

Before we estimate  $I_b$ , we need to make the following observation. The contour of integration in all the previous formulas as well as in the expression for  $I_b$ , can be bent towards the negative real axis. Namely, by  $\Gamma'(a)$ , we mean a union of two rays emanating from  $a$  and that make a  $\pm 45$  degree angle with the negative real axis. In all the preceeding formulas, the integration can be performed on either of the corresponding contours  $\Gamma$  or  $\Gamma'$  since the integrands are analytic and decay along the imaginary axis and decay exponentially in the negative real direction.

Then the change of variables  $\xi := \lambda t/T$  yields

$$I_b = \frac{1}{2\pi i} \frac{1}{T} \int_{\Gamma'(\frac{t}{T}(\gamma^2/2 + \delta))} \int_{|y| > \frac{\epsilon\sqrt{T}}{2}} e^{\xi - \sqrt{2\xi}\frac{|x-y|}{\sqrt{t}}} \frac{1}{2\pi|x-y|} (K(\lambda(\xi), T)f)(y) e^{y^2} e^{-y^2} dy d\xi.$$

Setting  $x = \sqrt{T}z$  and  $y = \sqrt{T}u$ , we get

$$|I_b| \leq \frac{C_1 \|f\|_{C_{\text{exp}}}}{(2\pi)^2} \frac{e^{-\frac{\epsilon^2}{2}T}}{T^{3/2}} \int_{\Gamma'(\frac{t}{T}(\gamma^2/2 + \delta))} \int_{|u| > \frac{\epsilon}{2}} \frac{|e^{\xi - \sqrt{2\xi}C_2|z-u|}}{|z-u|} e^{-(u^2 - \frac{\epsilon^2}{2})T} du d\xi,$$

from where the exponential decay of  $|I_b|$  as  $T \rightarrow \infty$  and thus the first claim of the Theorem follow.

It is not difficult to deduce (9) from (7) after noting that the fundamental solution at time  $t$  is the solution with the initial data  $p_\beta(t, y, \delta)$  evaluated at time  $t - \delta$ .  $\square$

*Proof of Proposition 2.3.* Let  $u_T = p_{\beta(T)} - p_0$ . Then

$$\frac{\partial}{\partial t} u_T = H_{\beta(T)} u_T + \beta v p_0,$$

with initial condition zero. By the Duhamel formula,

$$u_T(t, y, x) = \int_0^t \int_{\mathbb{R}^3} p_{\beta(T)}(t-s, z, x) \beta v(z) p_0(s, y, z) dz ds,$$

which can be written as

$$\begin{aligned} u_T(t, y, x) &= \\ &= \int_0^t \int_{\mathbb{R}^3} \frac{1}{\sqrt{T}} \frac{\kappa}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\lambda \frac{t-s}{T} - \sqrt{2\lambda} \frac{|x|}{\sqrt{T}}}}{(\sqrt{2\lambda} - \gamma)|x|} d\lambda \psi(z) \beta(T) v(z) p_0(s, y, 0) dz ds + h_T(t, y, x) \end{aligned}$$

where  $h_T(t, y, x)$  is an error term. The first term, denoted by  $u^0(t, y, x)$ , can be easily seen to equal

$$u_T^0(t, y, x) = \frac{1}{\sqrt{T}} \frac{\beta(T)}{\beta_{\text{cr}}} \int_0^t w_{T,\lambda,x}(t-s) p_0(s, y, 0) ds, \quad (22)$$

where

$$w_{T,\lambda,x}(t) = \frac{1}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\lambda \frac{t}{T} - \sqrt{2\lambda} \frac{|x|}{\sqrt{T}}}}{(\sqrt{2\lambda} - \gamma)|x|} d\lambda. \quad (23)$$

We can evaluate the convolution in (22) in the following way. First note that (23) is an inverse transform, while the transform of  $p_0(s, y, 0)$  is  $e^{-\sqrt{2\lambda}|y|}/2\pi|y|$ , and thus the transform of the convolution is

$$\frac{T}{2\pi} \frac{e^{-\sqrt{2\lambda}(|x|+|y|)}}{(\sqrt{2\lambda T} - \gamma)|x||y|}.$$

Applying the inverse formula of the Laplace transform and substituting  $\lambda \rightarrow \lambda/T$ , we get the main term in (10).

The remainder term can be written as

$$h_T(t, y, x) = h_T^{(1)}(t, y, x) + h_T^{(2)}(t, y, x),$$

where

$$\begin{aligned} h_T^{(1)}(T, t, y, x) &= \int_0^t \int_{\mathbb{R}^3} \frac{1}{\sqrt{T}} \frac{\kappa}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\lambda \frac{t}{T} - \sqrt{2\lambda} \frac{|x|}{\sqrt{T}}}}{(\sqrt{2\lambda} - \gamma)|x|} d\lambda \cdot \\ &\quad \cdot \psi(z) v(z) \beta(T) (p_0(s, y, z) - p_0(s, y, 0)) dz ds. \end{aligned}$$

Using the same Laplace transform trick, this can be shown to equal

$$\frac{1}{\sqrt{T}} \frac{\beta(T)}{\beta_{\text{cr}}} \frac{1}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\lambda \frac{t}{T}}}{\sqrt{2\lambda} - \gamma} \frac{e^{-\sqrt{2\lambda} \frac{|x|}{\sqrt{T}}}}{|x|} \int_{\mathbb{R}^3} \left( \frac{e^{-\sqrt{2\lambda} \frac{|y-z|}{\sqrt{T}}}}{|y-z|} - \frac{e^{-\sqrt{2\lambda} \frac{|y|}{\sqrt{T}}}}{|y|} \right) dz d\lambda,$$

and this can be shown to satisfy (11) the same way the main term in (7) followed from (21). The last remaining term  $h_T^{(2)}$  containing the error term of (9) can easily be shown to satisfy (11).  $\square$

*Proof of Proposition 2.4.* Applying the Laplace transform techniques, it is not difficult to show (see Lemma 7.1 in [3]) that

$$Z_{\beta(T),t}(x) - 1 = -\frac{1}{2\pi i} \int_{\Gamma(\lambda_0(\beta(T)) + \frac{\delta}{T})} \frac{e^{\lambda t}}{\lambda} (R_{\beta(T)}(\lambda)(\beta v))(x) d\lambda.$$

Using this formula, one can prove the claim following the same steps as in the above proof of Proposition 2.2.  $\square$

The next lemma easily follows from the Feynman-Kac formula.

**Lemma 3.6.** *For  $0 = t_0 < t_1 < \dots < t_n \leq T$ ,*

$$P_{\beta,T}(x(t_1) \in dx_1, \dots, x(t_n) \in dx_n) = \frac{\prod_{i=0}^{n-1} p_{\beta}(t_{i+1} - t_i, x_i, x_{i+1}) Z_{\beta,T-t_n}(x_n)}{Z_{\beta,T}} dx_1 \dots dx_n.$$

*Proof of Theorem 2.1.* By Lemma 3.6, the finite-dimensional densities of the measure  $f_{T^{-1}} P_{\beta(T),T}$  for  $0 < t_1 < \dots < t_n \leq 1$  and  $x_1, \dots, x_n \in \mathbb{R}^3$  are

$$\begin{aligned} \rho_{t_1, \dots, t_n}^T(x_1, \dots, x_n) &= \\ &= T^{3n/2} p_{\beta(T)}(t_1 T, 0, x_1 T^{1/2}) \dots p_{\beta(T)}((t_n - t_{n-1})T, x_{n-1} T^{1/2}, x_n T^{1/2}) \frac{Z_{\beta(T), T(1-t_n)}(x_n T^{1/2})}{Z_{\beta(T), T}(0)}. \end{aligned}$$

Let's introduce

$$p^T(s, t, y, x) = p_{\beta(T)}(T(t-s), y T^{1/2}, x T^{1/2})$$

and

$$R^T(s, t, y, x) = T^{3/2} \cdot \begin{cases} p_{\beta(T)}^T(s, t, y, x) \frac{Z_{\beta(T), T(1-t)}(x T^{1/2})}{Z_{\beta(T), T(1-s)}(y T^{1/2})} & t < 1, \\ \frac{p_{\beta(T)}^T(s, 1, y, x)}{Z_{\beta(T), T(1-s)}(y T^{1/2})} & t = 1. \end{cases}$$

Then it is not hard to show that

$$\rho_{t_1, \dots, t_n}^T(x_1, \dots, x_n) = R^T(0, t_1, 0, x_1) \cdot \dots \cdot R^T(t_{n-1}, t_n, x_{n-1}, x_n).$$

Note that by Proposition 2.2 and Proposition 2.3, we have for  $x \neq 0$

$$\lim_{T \rightarrow \infty} T p_{\beta(T)}(Tt, 0, x T^{1/2}) = \frac{\kappa \psi(0)}{2\pi i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\lambda t - \sqrt{2\lambda}|x|}}{(\sqrt{2\lambda} - \gamma)|x|} d\lambda, \quad (24)$$

while for  $x, y \neq 0$ ,

$$\begin{aligned} \lim_{T \rightarrow \infty} T^{3/2} p_{\beta(T)}(Tt, yT^{1/2}, xT^{1/2}) &= \\ &= \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{3/2}} + \frac{1}{4\pi^2 i} \int_{\Gamma(\frac{\gamma^2}{2} + \delta)} \frac{e^{\lambda t - \sqrt{2\lambda}(|x|+|y|)}}{(\sqrt{2\lambda} - \gamma)|x||y|} d\lambda. \end{aligned} \quad (25)$$

By Proposition 2.4, for  $x \neq 0$ ,

$$Z_{\beta(T), Tt}(xT^{1/2}) \rightarrow \overline{Z}_{\gamma, t}(x), \quad T \rightarrow \infty. \quad (26)$$

Using this, (25) and (4), it follows that for  $x, y \neq 0$ ,

$$\lim_{T \rightarrow \infty} R^T(s, t, y, x) = R(s, t, y, x) := \overline{p}_\gamma(t - s, y, x) \frac{\int_{\mathbb{R}^3} \overline{p}_\gamma(t, 1, x, z) dz}{\int_{\mathbb{R}^3} \overline{p}_\gamma(s, 1, y, z) dz}$$

Moreover, as follows from (24) and (4), it is not difficult to see that if  $x \neq 0$ ,

$$\lim_{T \rightarrow \infty} R^T(s, t, 0, x) = R(s, t, 0, x) := \lim_{y \rightarrow 0} \overline{p}_\gamma(t - s, y, x) \frac{\int_{\mathbb{R}^3} \overline{p}_\gamma(t, 1, x, z) dz}{\int_{\mathbb{R}^3} \overline{p}_\gamma(s, 1, y, z) dz}.$$

Since  $R(s, t, y, x)$  is the transition density of the polymer under the measure  $Q_\gamma$  (as discussed in Section 1 and as shown in [4]), this implies the convergence of the finite-dimensional distributions and the result will follow once tightness is shown. On the other hand, the proof of tightness is only a slight modification of the proof of Lemma 10.5 in [3] (where the case of fixed  $\beta$  was treated), so we don't provide it here.  $\square$

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